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SEARCH FOR A MOVING TARGET:

UPPER BOUND ON DETECTION PROBABILITY

by

Alan R. Washburn

October 1978

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## ABSTRACT

Existing algorithms for computing the optimal distribution of effort in search for a moving target operate by producing a sequence of progressively better distributions. This report shows how to compute an upper bound on the detection probability for each of those effort distributions in the case where the detection function is concave.

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## 1. INTRODUCTION

The object is to detect a randomly moving target at one of the discrete times $0,1,\ldots,\tau$. The searcher determines a non-negative effort distribution $\psi(x,t)$ such that the total effort applied at time t does not exceed $m(t)$. Our purpose here is to establish an upper bound on the detection probability for every effort distribution. This is intended to supplement existing iterative procedures that develop sequences of effort distributions that improve monotonically.

## 2. THE GENERAL CASE

Let $w(x,t)$ be an "effectiveness coefficient" for search effort applied at position $x$ at time t. If $X_t$ is the position of the target at time t, then the probability of detection depends, we assume, only on the total effective search effort

$$Z = \sum_{t=0}^{\tau} w(X_t,t)\ \psi(X_t,t) \ .$$

Specifically the probability of detection is $P(\psi) = E(b(Z))$, where the expectation operator is needed because $X_t$ is a stochastic process. We assume that $b(z') - b(z) \leq s(z)(z' - z)$ for some function $s(z)$ and for all $z'$; if the detection

function $b$ is concave and differentiable, then $s(z)$ is just $\frac{d}{dz} b(z)$. In most applications, $b(z) = 1 - \exp(-z)$ and $s(z) = \exp(-z)$.

Let $\psi$ and $\psi'$ be two effort distributions. Then

$$(1) \qquad P(\psi') - P(\psi) = E(b(Z') - b(Z)) \leq E(s(Z)\ (Z'-Z))$$

We also have

$$(2) \qquad E(s(Z)(Z'-Z)) = \sum_{t=0}^{\tau} E(s(Z)\ w(X_t,t)\ [\psi'(X_t,t) - \psi(X_t,t)])$$

We now consider two cases, depending on whether $X_t$ is discrete or continuous. If $X_t$ is discrete, we let $p_t(x)$ be the probability mass function of $X_t$, and require

$$\sum_x \psi'(x,t) = \sum_x \psi(x,t) = m(t)\ .$$

We define

$$(3) \qquad D_\tau(\psi,x,t) = w(x,t)\ p_t(x)\ E(s(Z)\,|\,X_t=x)$$

Then (2) can be written

$$(4) \qquad E(s(Z)(Z'-Z)) = \sum_{t=0}^{\tau} \sum_x D_\tau(\psi,x,t)\ [\psi'(x,t) - \psi(x,t)]$$

If $X_t$ is continuous, let $p_t(x)$ be the probability density function of $X_t$, and require $\int \psi'(x,t)dx = \int \psi(x,t)dx = m(t)$. Then (2) can be written

$$(4') \quad E(s(Z)(Z'-Z)) = \sum_{t=0}^{\tau} \int D_\tau(\psi,x,t) \; [\psi'(x,t) - \psi(x,t)]dx \;.$$

In either case, suppose that $D_\tau(\psi,x,t) \leq \bar{\lambda}(t)$ for all x, and $D_\tau(\psi,x,t) \geq \underline{\lambda}(t)$ when $\psi(x,t) > 0$. In the discrete case, from (4),

$$(5) \quad E(s(Z)(Z'-Z)) \leq \sum_{t=0}^{\tau} \; [\sum_x \bar{\lambda}(t)\psi'(x,t) - \sum_x \underline{\lambda} \; \psi(x,t)]$$

$$= \sum_{t=0}^{\tau} \; (\bar{\lambda}(t) - \underline{\lambda}(t)) \; m(t)$$

Combining (5) and (1),

$$(6) \qquad P(\psi') - P(\psi) \leq \sum_{t=0}^{\tau} \; (\bar{\lambda}(t) - \underline{\lambda}(t)) \; m(t) \; \equiv \Delta(\psi)$$

A formula similar to (5) shows that (6) must also hold in the continuous case.

Now $\psi'$ is not needed to compute any of the quantities on the right-hand side of (6), so every effort density $\psi'$ has associated with it an upper bound on the detection probability $P(\psi) + \Delta(\psi)$. In the event $\bar{\lambda}(t) = \underline{\lambda}(t)$ for all t, $\psi$ must actually be optimal--this has been observed by Stone [1], and in fact our whole development is a modification of his

sufficiency proof. The main issue is now computational: is determination of $\Delta(\psi)$ worth the effort?

## 3. THE CASE OF MARKOV MOTION AND EXPONENTIAL DETECTION FUNCTION.

Let

$$Z = Z_t^- + w(X_t,t)\ \psi(X_t,t) + Z_t^+ \qquad \text{for} \quad 0 \leq t \leq \tau,$$

where

$$Z_t^- = \sum_{u=0}^{t-1} w(X_u,u)\ \psi(X_u,u),$$

$$Z_t^+ = \sum_{u=t+1}^{\tau} w(X_u,u)\ \psi(X_u,u),$$

and

$$Z_0^- = Z_\tau^+ = 0 .$$

Then

$$D_\tau(\psi,x,t) = w(x,t)\ p_t(x)\ E(s(Z_t^- + w(x,t)\ \psi(x,t) + Z_t^+)\,|\,X_t=x) .$$

If the motion is Markov, then $Z_t^+$ is independent of $Z_t^-$ when $X_t = x$ is given. If $b(z) = 1 - \exp(-z)$, then $s(z) = \exp(-z)$. If both conditions hold, then

$$D_\tau(\psi,x,t) = w(x,t)\ P(\psi,x,t)\ \exp(-w(x,t)\ \psi(x,t))\ Q(\psi,x,t) ,$$

where

$$P(\psi,x,t) \equiv p_t(x)\ E(s(Z_t^-)\,|\,X_t=x)$$

and

$$Q(\psi,x,t) \equiv E(s(Z_t^+)\,|\,X_t=x) .$$

$P(\psi,x,t)$ is the joint probability that $X_t = x$ and that the target is not detected by any of the searches at $0,1,\ldots,t-1$; note that $P(\psi,x,t)$ does not depend on $\psi(y,u)$ for $u \geq t$. $Q(\psi,x,t)$ is the conditional probability that the target is not detected by any of the searches at $t+1,\ldots,\tau$ given that $X_t = x$; note that $Q(\psi,x,t)$ does not depend on $\psi(y,u)$ for $u \leq t$. Given $\psi$, $P(\psi,x,t)$ and $Q(\psi,x,t)$ can be easily computed recursively. $P(\psi,x,0)$ is a given initial distribution, and $P(\psi,x,t+1)$ can be obtained from $P(\psi,x,t)$ using the Markov transition rule and $\psi(\cdot,t)$. Similarly, $Q(\psi,x,\tau) \equiv 1$, and $Q(\psi,x,t-1)$ can be obtained from $Q(\psi,x,t)$ using the Markov transition rule and $\psi(\cdot,t)$. After obtaining $P(\psi,x,t)$ and $Q(\psi,x,t)$, it is a simple matter to compute $D_\tau(\psi,x,t)$ and then $\Delta(\psi)$.

Algorithms for finding the optimal effort distribution $\psi^*$ typically operate by generating a sequence $\psi_1, \psi_2, \ldots$ that approaches $\psi^*$. The method of computation is such that $\Delta(\psi_i)$ can be computed with only slightly more effort. Consider the discrete case. For any $t$, the probability of detection is

$$(7) \qquad P(\psi) = 1 - \sum_x P(\psi,x,t) \exp(-w(x,t)\ \psi(x,t))\ Q(\psi,x,t)$$

In order to find $\psi^*$, we first make an initial guess $Q^0(x,t)$ and then calculate $\psi^n$ the function that minimizes

$$\sum_x P(\psi,x,t)\, \exp(-w(x,t)\, \psi(x,t))\, Q^{n-1}(x,t) \qquad \text{for} \quad n \geq 1,$$

with $Q^n(x,t) = Q(\psi^n,x,t)$ for $n \geq 1$. Each of the minimization problems is relatively simple, and it can be shown that $P(\psi^n)$ increases with $n$ [1,2,3].

For each of the minimization problems, there must exist a function $\lambda_n(t)$ such that $D_n(x,t) = \lambda_n(t)$ when $\psi^n(x,t) > 0$ and $D_n(x,t) \leq \lambda_n(t)$ when $\psi^n(x,t) = 0$, where

$$D_n(x,t) = w(x,t)\, P(\psi^n,x,t)\, \exp(-w(x,t)\, \psi^n(x,t))\, Q^{n-1}(x,t).$$

These are simply the Kuhn-Tucker conditions; in practice, some sort of a search is made until $\lambda_n(t)$ is found such that $\sum_x \psi(x,t) = m(t)$. But

$$(8) \quad D_\tau(\psi^n,x,t) \begin{cases} = \lambda_n(t)\, Q^n(x,t)/Q^{n-1}(x,t) & \text{when} \quad \psi^n(x,t) > 0 \\ \leq \lambda_n(t)\, Q^n(x,t)/Q^{n-1}(x,t) & \text{when} \quad \psi^n(x,t) = 0 \end{cases}$$

so computation of $D_\tau(\psi^n,x,t)$ is a small burden given that $\lambda_n(t)$, $Q^n(x,t)$, and $Q^{n-1}(x,t)$ have to be computed in any case.

In the continuous case, $\sum_x$ is replaced by $\int dx$; otherwise, the development remains the same.

## 4. AN EXAMPLE

We set $\tau = 79$, so there are 80 looks. The target does a random walk over the cells 1,.., 67, starting at cell 34. At each opportunity, it moves left with probability .3, right with probability .3, or else does not move. The boundary is reflecting. We set $w(x,t) = .001625$ and $m(t) = 100$; if all 100 units of search effort are used in a single cell, the probability of detection would be $1 - \exp(.1625) = .15$ at each of the 80 opportunities. We also set $Q^0(x,t) \equiv 1$.

The first five detection probabilities (and values for $\Delta(\psi)$) are: .7054 (.0393), .7075 (.0246), .7086 (.0120), .7090 (.0067), and .7092 (.0036). The amounts of CPU time required on the NPS IBM 360/67 for computation of each of these five pairs were 2.4, 2.8, 2.4, 2.0, and 2.0 seconds, respectively. The second pass always takes a relatively long time because $Q^1(x,t)$ is radically different from $Q^0(x,t)$. The function $\exp(-w(x,t)\ \psi^5(x,t))$ is shown to three decimal places in Figure 1, with **** indicating no search. Time reads down the page (there are 80 rows). Only the central cells are shown; the rest are not searched.

It is interesting to compare these results with those that result when all effort must be placed in a single cell at each time. The best probability of detection in that case is apparently .64--sufficient conditions for optimality are not known in that case, but the author has done sufficient

experimentation to conjecture that .64 is the answer. The increase from .64 to .7092 can, of course, be attributed to the relaxation of a constraint.

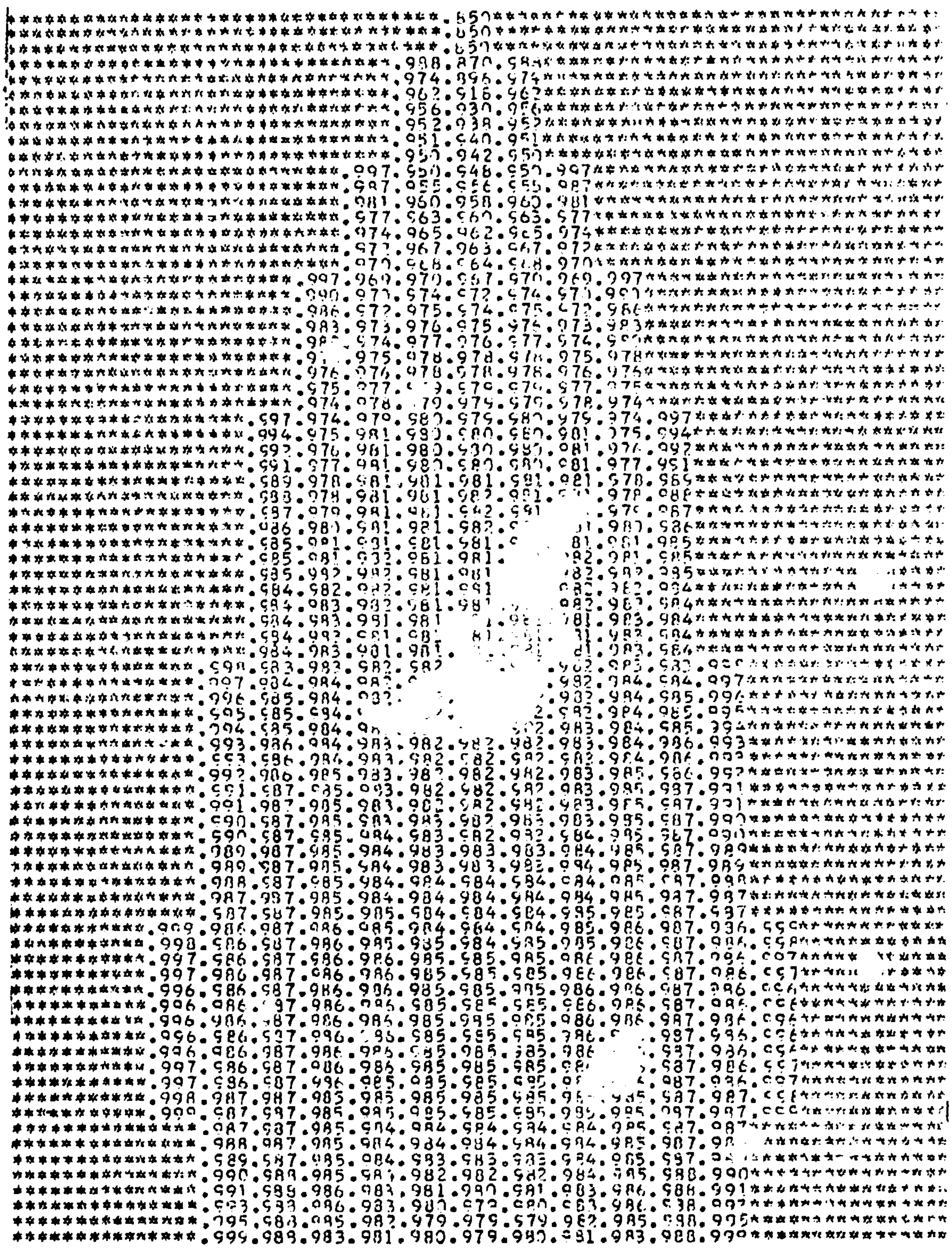
## REFERENCES


(1) "Numerical Optimization of Search for a Moving Target," Daniel H. Wagner, Associates Report to Office of Naval Research, by L. D. Stone et al., 23 June 1978.

(2) "Optimal Search for a Moving Target in Discrete Time and Space," by S. S. Brown (submitted to Opns. Res.).

(3) "On Search for a Moving Target," by A. R. Washburn (submitted to NRLQ).

FIGURE 1. Survival Probabilities for Near Optimal Search.



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